

Divergence form nonlinear nonsmooth parabolic equations with locally arbitrary growth conditions and nonlinear maximal regularity

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Abstract

This is a generalization of our prior work on the compact fixed point theory for the elliptic Rosseland-type equations. Inspired by the Rosseland equation in the conduction-radiation coupled heat transfer, we use the locally arbitrary growth conditions instead of the common global restricted growth conditions. Its physical meaning is: the absolute temperature should be positive and bounded.

There exists a fixed point for the linearized map (compact and continuous in L^2) in a closed convex set. We also consider the nonlinear maximal regularity in Sobolev space.

Key words: arbitrary growth conditions; fixed point;
Rosseland equation; nonlinear maximal regularity;
nonlinear parabolic equations; nonsmooth data.

1 Introduction

Suppose $S = (0, T)$ where T is a positive constant. Consider the following parabolic problem:

$$\partial_t u - \operatorname{div} [A(u(x), x, t) \nabla u] = 0, \quad \text{in } Q_T = \Omega \times S. \quad (1.1)$$

The weak solution can be defined as the following: find u ,

$$(u - g) \in L^2(S; H_0^1(\Omega)), \quad (u - g)(x, 0) = 0, \quad (1.2)$$

(so we know the boundary and initial conditions)

$$\partial_t u \in L^2(S; H^{-1}(\Omega)), \quad (1.3)$$

where $L^2(S; H^{-1}(\Omega))$ is the dual space of $L^2(S; H_0^1(\Omega))$, such that $\forall \varphi \in L^2(S; H_0^1(\Omega))$,

$$\langle \partial_t u, \varphi \rangle_{L^2(S; H_0^1(\Omega))} + \iint_{Q_T} A(u(x), x) \nabla u \cdot \nabla \varphi = 0. \quad (1.4)$$

For the definitions of these spaces, see [1, 4].

For the Rosseland equation: $A(z, x, t) = K(x, t) + z^3 B(x, t)$, where $K(x, t)$ and $B(x, t)$ are symmetric and positive definite.

(1) $K(x, t) + z^3 B(x, t)$ is positive definite only in an interval for z .

(2) it doesn't satisfy the common growth and smooth conditions and there may be no $C^{2,\gamma}$ estimate (Theorem 15.11 [6]).

The problem of the existence theory for the Rosseland equation (also named diffusion approximation) was proposed by Laitinen [11] in 2002. It may be useful to keep this equation in mind while reading this paper.

It's a little technical to prove the existence by the fixed point method in $L^\infty(Q_T)$ (or $C^0(\overline{Q_T})$) [12, 13]. We will use $L^2(Q_T)$ in this paper.

Firstly, we make the following assumptions.

(A1) $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. $S = (0, T)$ where T is a positive constant. $Q_T = \Omega \times S$.

(A2) $A = (a_{ij})$. $a_{ij} = a_{ji}$. $T_{min} \leq T_{max}$ are two constants.

$$\lambda |\xi|^2 \leq a_{ij}(z, x, t) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad 0 < \lambda \leq \Lambda, \quad (1.5)$$

$$\forall (z, x, t, \xi) \in [T_{min}, T_{max}] \times Q_T \times \mathbb{R}^n. \quad (1.6)$$

Here we use the Einstein summation convention.

(A3) $\partial_p Q_T = \{\partial \Omega \times S\} \cup \{(x, 0); x \in \Omega\}$,

$$g \in H^1(Q_T). \quad T_{min} \leq g(x, t) \leq T_{max}, \quad \text{a.e. in } \partial_p Q_T. \quad (1.7)$$

(A4) $A(z, x, t)$ is uniformly continuous with respect to z in \mathfrak{C} , where

$$\mathfrak{C} = \{\varphi \in L^2(Q_T); T_{min} \leq \varphi(x, t) \leq T_{max}, \text{ a.e. in } Q_T\}. \quad (1.8)$$

In other words, if $z_i, z \in \mathfrak{C}$, $\|z_i - z\|_2 \rightarrow 0$,

$$\sup_{1 \leq p, q \leq n} \|a_{pq}(z_i(x, t), x, t) - a_{pq}(z(x, t), x, t)\|_2 \rightarrow 0. \quad (1.9)$$

Remark 1.1 In fact, we had considered a general case: parabolic equations with $(c(x)\rho(x)u)'$, nonnegative bounded mixed boundary conditions and right-hand term $f(z, x, t)$ [13].

If a_{pq} is uniformly Hölder continuous with respect to z , (A4) is natural since

$$\begin{aligned} \|a_{pq}(z_i(x, t), x, t) - a_{pq}(z(x, t), x, t)\|_2^2 &\leq \iint_{Q_T} C |z_i(x, t) - z(x, t)|^{2\alpha} \\ &\leq C \|z_i - z\|_2^2 \rightarrow 0. \end{aligned} \quad (1.10)$$

Theorem 1.1 (Parabolic spaces) *Let*

$$W \equiv \{w \in L^2(S; H_0^1(\Omega)); \partial_t w \in L^2(S; H^{-1}(\Omega))\}, \quad (1.11)$$

$$\|w\|_W^2 = \|w\|_{L^2(S; H_0^1(\Omega))}^2 + \|\partial_t w\|_{L^2(S; H^{-1}(\Omega))}^2, \quad (1.12)$$

then (1) (page 173 [1], page 61 [4])

$$W \hookrightarrow C([0, T]; L^2(\Omega)), \quad W \hookrightarrow L^2(Q_T). \quad (1.13)$$

The last imbedding is compact.

(2) (page 173 [1]) $C^\infty([0, T]; H_0^1(\Omega))$ is dense in W .

(3) (Theorem 1.6 [8]) *Let*

$$W_{cp} \equiv \{w \in L^2(S; X); \partial_t [c\rho(x)w] \in L^2(S; X')\}, \quad (1.14)$$

then $C^\infty([0, T]; X)$ is dense in W_{cp} . For the mixed boundary conditions, we can let $X = H_D^1(\Omega)$.

Theorem 1.2 ($V_2^{1,0}(Q_T)$) (page 42-44 [2]) *Let*

$$V_2^{1,0}(Q_T) \equiv L^2(S; H^1(\Omega)) \cap C([0, T]; L^2(\Omega)), \quad (1.15)$$

$$\|w\|_{V_2^{1,0}(Q_T)}^2 = \|\nabla w\|_{L^2(Q_T; \mathbb{R}^n)}^2 + \sup_{t \in [0, T]} \|w(x, t)\|_{L^2(\Omega)}^2, \quad (1.16)$$

then

(1) $H^1(Q_T) \subset V_2^{1,0}(Q_T)$.

(2) If $u(x, t) \in V_2^{1,0}(Q_T)$, $\forall k \in \mathbb{R}$,

$$(u - k)_+(x, t) = \max\{(u - k)(x, t), 0\} \in V_2^{1,0}(Q_T). \quad (1.17)$$

(3) If $\|u_i - u\|_{V_2^{1,0}(Q_T)} \rightarrow 0$, then $\forall k \in \mathbb{R}$,

$$\|(u_i - k)_+ - (u - k)_+\|_{V_2^{1,0}(Q_T)} \rightarrow 0. \quad (1.18)$$

2 Linearized map and fixed point

Theorem 2.1 (Corollary 11.2 [6]) *Let \mathfrak{C} be a closed convex set in a Banach space \mathfrak{B} and let \mathcal{L} be a continuous mapping of \mathfrak{C} into itself such that the image $\mathcal{L}\mathfrak{C}$ is precompact. Then \mathcal{L} has a fixed point.*

Lemma 2.1 *The following set*

$$\mathfrak{C} = \{\varphi \in L^2(Q_T); T_{\min} \leq \varphi(x, t) \leq T_{\max}, \text{ a. e. in } Q_T\}. \quad (2.19)$$

is a closed convex set in the Banach space $L^2(Q_T)$.

Proof Suppose $v_i \in \mathfrak{C}$, $v \in L^2(Q_T)$, $\|v_i - v\|_2 \rightarrow 0$. If $v \notin \mathfrak{C}$, there exist two constants $\delta_0 > 0$, $\delta_1 > 0$, such that the Lebesgue measure of the set $Q_0 \equiv \{(x, t) \in Q_T; v(x, t) \geq T_{\max} + \delta_0\}$ is bigger than $\delta_1 > 0$. Then

$$\|v_i - v\|_2^2 = \iint_{Q_T} |v_i - v|^2 \geq \iint_{Q_0} |v_i - v|^2 \geq \delta_0^2 \delta_1. \quad (2.20)$$

It's impossible since $\|v_i - v\|_2 \rightarrow 0$. Similarly, $v \geq T_{\min}$ and \mathfrak{C} is closed.

$$\forall \theta \in [0, 1], \quad \theta v_1 + (1 - \theta)v_2 \leq \theta T_{\max} + (1 - \theta)T_{\max} = T_{\max}. \quad (2.21)$$

So \mathfrak{C} is convex. \square

Theorem 2.2 *If (A1) – (A4) are satisfied, then*

(1) *$\forall z \in \mathfrak{C}$, there exists a unique w ,*

$$(w - g) \in L^2(S; H_0^1(\Omega)), \quad (w - g)(x, 0) = 0, \quad (2.22)$$

$$w \in \mathfrak{C}, \quad \partial_t w \in L^2(S; H^{-1}(\Omega)), \quad (2.23)$$

such that $\forall \varphi \in L^2(S; H_0^1(\Omega))$,

$$\langle \partial_t w, \varphi \rangle_{L^2(S; H_0^1(\Omega))} + \iint_{Q_T} A(z(x, t), x, t) \nabla w \cdot \nabla \varphi = 0. \quad (2.24)$$

(2) *Define a map $\mathcal{L} : \mathfrak{C} \rightarrow \mathfrak{C}$, $\mathcal{L}z = w$, then $\mathcal{L}\mathfrak{C}$ is precompact in $L^2(Q_T)$.*

(3) *\mathcal{L} is continuous in $L^2(Q_T)$. So \mathcal{L} has a fixed point in \mathfrak{C} .*

Proof (1) For the *a priori* estimate, since $H^{-1}(\Omega) \hookrightarrow L^2(\Omega)$ (page 55, 60 [4]),

$$\partial_t g \in L^2(Q_T) = L^2(S; L^2(\Omega)) \hookrightarrow L^2(S; H^{-1}(\Omega)), \quad (2.25)$$

$$v \equiv (w - g) \in W \hookrightarrow C([0, T]; L^2(\Omega)), \quad (2.26)$$

$$g \in H^1(Q_T) \hookrightarrow C([0, T]; L^2(\Omega)), \quad w \in C([0, T]; L^2(\Omega)). \quad (2.27)$$

$$w \in L^2(S; H^1(\Omega)), \quad w \in V_2^{1,0}(Q_T). \quad (2.28)$$

Let

$$\varphi = (w - T_{max})_+ \in V_2^{1,0}(Q_T), \quad (2.29)$$

then

$$\varphi(x, t)|_{\partial_p Q_T} = 0, \quad \varphi \in L^2(S; H_0^1(\Omega)). \quad (2.30)$$

For any $v_i \in C^\infty([0, T]; H_0^1(\Omega)) \subset H^1(Q_T)$, we have

$$\begin{aligned} & \langle \partial_t(v_i + g), (v_i + g - T_{max})_+ \rangle_{L^2(S; H_0^1(\Omega))} \\ &= \iint_{Q_T} \partial_t(v_i + g) \cdot (v_i + g - T_{max})_+ \\ &= \iint_{Q_T} \partial_t \frac{(v_i + g - T_{max})_+(x, t)^2}{2} \\ &= \int_{\Omega} \frac{(v_i + g - T_{max})_+(x, T)^2}{2} \\ & \quad - \int_{\Omega} \frac{(v_i + g - T_{max})_+(x, 0)^2}{2}. \end{aligned} \quad (2.31)$$

By the density of $C^\infty([0, T]; H_0^1(\Omega))$ in W , for $v \equiv (w - g) \in W$, we can find $\{v_i\} \subset C^\infty([0, T]; H_0^1(\Omega))$ such that

$$\|v_i - v\|_{C([0, T]; L^2(\Omega))} \leq C\|v_i - v\|_W \rightarrow 0, \quad (2.32)$$

$$\|(v_i + g) - (v + g)\|_{V_2^{1,0}(Q_T)} = \|v_i - v\|_{V_2^{1,0}(Q_T)} \rightarrow 0, \quad (2.33)$$

$$\|(v_i + g - T_{max})_+ - (v + g - T_{max})_+\|_{V_2^{1,0}(Q_T)} \rightarrow 0. \quad (2.34)$$

$$\|(v_i + g - T_{max})_+ - (v + g - T_{max})_+\|_{L^2(S; H^1(\Omega))} \rightarrow 0. \quad (2.35)$$

$$v_i, v|_{\partial\Omega \times S} = 0, \quad g|_{\partial\Omega \times S} \leq T_{max}. \quad (2.36)$$

$$\|(v_i + g - T_{max})_+ - (v + g - T_{max})_+\|_{L^2(S; H_0^1(\Omega))} \rightarrow 0. \quad (2.37)$$

$$\|\partial_t(v_i + g) - \partial_t(v + g)\|_{L^2(S; H^{-1}(\Omega))} \rightarrow 0, \quad (2.38)$$

$$\begin{aligned}
& \int_{\Omega} [(v_i + g - T_{max})_+(x, t)^2 - (v + g - T_{max})_+(x, t)^2] \\
= & \int_{\Omega} [(v_i + g - T_{max})_+(x, t) + (v + g - T_{max})_+(x, t)] \\
& [(v_i + g - T_{max})_+(x, t) - (v + g - T_{max})_+(x, t)] \\
\leq & \| (v_i + g - T_{max})_+(x, t) + (v + g - T_{max})_+(x, t) \|_{L^2(\Omega)} \\
& \| (v_i + g - T_{max})_+(x, t) - (v + g - T_{max})_+(x, t) \|_{L^2(\Omega)} \\
\leq & (\| (v_i + g - T_{max})_+(x, t) \|_{L^2(\Omega)} + \| (v + g - T_{max})_+(x, t) \|_{L^2(\Omega)}) \\
& \| (v_i + g - T_{max})_+(x, t) - (v + g - T_{max})_+(x, t) \|_{L^2(\Omega)} \\
\leq & (\| (v_i + g - T_{max})_+(x, s) \|_{C([0, T]; L^2(\Omega))} \\
& + \| (v + g - T_{max})_+(x, s) \|_{C([0, T]; L^2(\Omega))}) \| v_i - v \|_{C([0, T]; L^2(\Omega))} \\
\leq & C \| v_i - v \|_W \rightarrow 0. \tag{2.39}
\end{aligned}$$

$$\begin{aligned}
& \langle \partial_t w, (w - T_{max})_+ \rangle_{L^2(S; H_0^1(\Omega))} \\
= & \langle \partial_t (v + g), (v + g - T_{max})_+ \rangle_{L^2(S; H_0^1(\Omega))} \\
= & \lim_{i \rightarrow \infty} \langle \partial_t (v_i + g), (v_i + g - T_{max})_+ \rangle_{L^2(S; H_0^1(\Omega))} \\
= & \lim_{i \rightarrow \infty} \int_{\Omega} \left[\frac{(v_i + g - T_{max})_+(x, T)^2}{2} - \frac{(v_i + g - T_{max})_+(x, 0)^2}{2} \right] \\
= & \int_{\Omega} \frac{(v + g - T_{max})_+(x, T)^2}{2} - \int_{\Omega} \frac{(v + g - T_{max})_+(x, 0)^2}{2} \\
= & \int_{\Omega} \frac{(v + g - T_{max})_+(x, T)^2}{2} \geq 0. \tag{2.40}
\end{aligned}$$

$$\begin{aligned}
& \iint_{Q_T} A(z(x, t), x, t) \nabla w \cdot \nabla (w - T_{max})_+ \\
= & \iint_{Q_T} A(z(x, t), x, t) \nabla (w - T_{max})_+ \cdot \nabla (w - T_{max})_+ \\
\geq & \lambda \int_S \int_{\Omega} |\nabla (w - T_{max})_+|^2 \\
\geq & C(\Omega) \lambda \int_S \int_{\Omega} (w - T_{max})_+^2. \tag{2.41}
\end{aligned}$$

$$\begin{aligned}
& C(\Omega) \lambda \int_S \int_{\Omega} (w - T_{max})_+^2 \\
\leq & \langle \partial_t w, (w - T_{max})_+ \rangle_{L^2(S; H_0^1(\Omega))} \\
& + \iint_{Q_T} A(z(x, t), x, t) \nabla w \cdot \nabla (w - T_{max})_+ \\
= & 0. \tag{2.42}
\end{aligned}$$

So $w \leq T_{max}$, a. e. in Q_T . Similarly, $w \in \mathfrak{C}$.

For the well-posedness (the existence, uniqueness and the estimate in W) of w , $(w - g) \in W$, we refer to Galerkin method (page 171 [1], page 77 [3], page 205-211 [4]; for mixed problems, see Theorem 2.2 [8]).

(2) $\|(w - g)\|_W \leq C$. W can be compactly imbedded in $L^2(Q_T)$, so $\mathcal{L}\mathfrak{C}$ is precompact in $L^2(Q_T)$.

(3) Suppose

$$z_i, z \in \mathfrak{C}, \quad \|z_i - z\|_2 \rightarrow 0, \quad \mathcal{L}z_i = w_i, \quad \mathcal{L}z = w. \quad (2.43)$$

W is a Hilbert and thus a reflexive space, so there exists a subsequence $\{i_k\}$ and $v_0 = (w_0 - g) \in W$ such that

$$(w_{i_k} - g) \rightarrow (w_0 - g), \quad \text{weakly in } W. \quad (2.44)$$

$$W \subset L^2(S; H_0^1(\Omega)), \quad (L^2(S; H_0^1(\Omega)))' \subset W'. \quad (2.45)$$

$$(w_{i_k} - g) \rightarrow (w_0 - g), \quad \text{weakly in } L^2(S; H_0^1(\Omega)). \quad (2.46)$$

$$\nabla(w_{i_k} - g) \rightarrow \nabla(w_0 - g), \quad \text{weakly in } L^2(Q_T; \mathbb{R}^n). \quad (2.47)$$

$$\nabla w_{i_k} \rightarrow \nabla w_0, \quad \text{weakly in } L^2(Q_T; \mathbb{R}^n). \quad (2.48)$$

$$\|w_{i_k} - g - w_0 + g\|_2 \rightarrow 0, \quad \|w_{i_k} - w_0\|_2 \rightarrow 0. \quad (2.49)$$

$$\partial_t(w_{i_k} - g) \rightarrow \partial_t(w_0 - g), \quad \text{weakly in } L^2(S; H^{-1}(\Omega)). \quad (2.50)$$

$$\partial_t g \in L^2(Q_T) \subset L^2(S; H^{-1}(\Omega)). \quad (2.51)$$

$$\partial_t w_{i_k} \rightarrow \partial_t w_0, \quad \text{weakly in } L^2(S; H^{-1}(\Omega)). \quad (2.52)$$

$\forall \phi \in C^\infty([0, T]; C_0^\infty(\Omega))$, using the natural map into its second dual (page 89 [5]),

$$\begin{aligned} & \langle F(\phi), \partial_t w_{i_k} - \partial_t w_0 \rangle_{L^2(S; H^{-1}(\Omega))} \\ & \equiv \langle \partial_t w_{i_k} - \partial_t w_0, \phi \rangle_{L^2(S; H_0^1(\Omega))}. \end{aligned} \quad (2.53)$$

$$\begin{aligned} & \langle F(\phi), \partial_t w_{i_k} - \partial_t w_0 \rangle_{L^2(S; H^{-1}(\Omega))} \rightarrow 0, \\ \Rightarrow & \langle \partial_t w_{i_k} - \partial_t w_0, \phi \rangle_{L^2(S; H_0^1(\Omega))} \rightarrow 0. \end{aligned} \quad (2.54)$$

$$\begin{aligned}
& \left| \iint_{Q_T} [A(z_{i_k}(x, t), x, t) \nabla w_{i_k} - A(z(x, t), x, t) \nabla w_0] \cdot \nabla \phi \right| \\
\leq & \left| \iint_{Q_T} [A(z_{i_k}(x, t), x, t) \nabla w_{i_k} - A(z(x, t), x, t) \nabla w_{i_k}] \cdot \nabla \phi \right| \\
& + \left| \iint_{Q_T} [A(z(x, t), x, t) \nabla w_{i_k} - A(z(x, t), x, t) \nabla w_0] \cdot \nabla \phi \right| \\
= & \left| \iint_{Q_T} [A(z_{i_k}(x, t), x, t) - A(z(x, t), x, t)] \nabla w_{i_k} \cdot \nabla \phi \right| \\
& + \left| \iint_{Q_T} [\nabla w_{i_k} - \nabla w_0] \cdot A(z(x, t), x, t)^\top \nabla \phi \right| \\
\leq & C \sup_{1 \leq p, q \leq n} \|a_{pq}(z_{i_k}(x, t), x, t) - a_{pq}(z(x, t), x, t)\|_2 + \epsilon(i_k) \\
\rightarrow & 0. \tag{2.55}
\end{aligned}$$

$$\begin{aligned}
& \langle \partial_t w_{i_k}, \phi \rangle_{L^2(S; H_0^1(\Omega))} \\
& + \iint_{Q_T} A(z_{i_k}(x, t), x, t) \nabla w_{i_k} \cdot \nabla \phi = 0. \tag{2.56}
\end{aligned}$$

$$\begin{aligned}
& \langle \partial_t w_0, \phi \rangle_{L^2(S; H_0^1(\Omega))} \\
& + \iint_{Q_T} A(z(x, t), x, t) \nabla w_0 \cdot \nabla \phi = 0. \tag{2.57}
\end{aligned}$$

From the density of $C^\infty([0, T]; C_0^\infty(\Omega))$ in $L^2(S; H_0^1(\Omega))$, $\forall \varphi \in L^2(S; H_0^1(\Omega))$,

$$\begin{aligned}
& \langle \partial_t w_0, \varphi \rangle_{L^2(S; H_0^1(\Omega))} \\
& + \iint_{Q_T} A(z(x, t), x, t) \nabla w_0 \cdot \nabla \varphi = 0. \tag{2.58}
\end{aligned}$$

For the boundary condition,

$$(w_0 - g) \in L^2(S; H_0^1(\Omega)), \quad (w_0 - g)|_{\partial\Omega \times S} = 0. \tag{2.59}$$

For the initial condition, $\forall \psi(x) \in L^2(\Omega)$, we can define a linear functional on W ,

$$\langle \Psi, h \rangle_W \equiv \int_{\Omega} h(x, 0) \psi, \quad \forall h(x, t) \in W. \tag{2.60}$$

This functional is bounded since

$$\begin{aligned}
|\langle \Psi, h \rangle_W| &= \left| \int_{\Omega} h(x, 0) \psi \right| \\
&\leq \|h(x, 0)\|_2 \|\psi\|_2 \leq \|\psi\|_2 \sup_{s \in [0, T]} \|h(x, s)\|_2 \\
&= \|\psi\|_2 \|h(x, t)\|_{C([0, T]; L^2(\Omega))} \\
&\leq C \|h(x, t)\|_W.
\end{aligned} \tag{2.61}$$

Since

$$(w_{i_k} - g) \rightarrow (w_0 - g), \quad \text{weakly in } W, \tag{2.62}$$

$\forall \psi(x) \in L^2(\Omega),$

$$\langle \Psi, (w_{i_k} - g) - (w_0 - g) \rangle_W \equiv \int_{\Omega} [(w_{i_k} - g) - (w_0 - g)](x, 0) \psi \rightarrow 0. \tag{2.63}$$

From the Riesz Representation Theorem in $L^2(\Omega)$,

$$(w_{i_k} - g)(x, 0) \rightarrow (w_0 - g)(x, 0), \quad \text{weakly in } L^2(\Omega). \tag{2.64}$$

Note that from the initial condition,

$$(w_{i_k} - g)(x, 0) = 0, \quad \text{in } L^2(\Omega). \tag{2.65}$$

$$(w_{i_k} - g)(x, 0) \rightarrow 0, \quad \text{strongly in } L^2(\Omega). \tag{2.66}$$

$$(w_0 - g)(x, 0) = 0, \quad \text{in } L^2(\Omega). \tag{2.67}$$

To sum up, $w_0 = \mathcal{L}z$: w_0 satisfies the linearized equation and the initial-boundary conditions.

Since the solution is unique from the step (1), $w_0 = \mathcal{L}z = w$. So $\|w_{i_k} - w\|_2 \rightarrow 0$. Each subsequence of $\{\|w_i - w\|_2\}$ has a sub-subsequence which converges to 0, so $\|w_i - w\|_2 \rightarrow 0$. We have obtain the continuity of \mathcal{L} .

From Theorem 2.1, there exists a fixed point. \square

Remark 2.1 For the continuity of \mathcal{L} in $C^0(\overline{Q}_T)$, we can use the well-known De Giorgi-Nash estimate: $\{w_i\}$ is bounded in $C^{2\alpha, \alpha}(\overline{Q}_T)$ if $g \in C^{2\alpha_0, \alpha_0}(\partial_p Q_T)$ and Ω is an (A) domain (page 145 [2]).

Then from the Arzelà-Ascoli Lemma, $\|w_{i_k} - w_0\|_{C^0(\overline{Q}_T)} \rightarrow 0$. By the same method, $w_0 = w$ and $\|w_i - w\|_{C^0(\overline{Q}_T)} \rightarrow 0$.

From the linear maximal regularity [7, 8], a natural conjecture is: \mathcal{L} is continuous in $C^{2\alpha, \alpha}(\overline{Q}_T)$ and W .

3 Nonlinear maximal regularity

For the linear parabolic/elliptic equations with nonsmooth data, the theory of maximal regularity has been established [7, 8, 9, 10]. In brief, maximal regularity is about the smoothness of the data-to-solution-map [10]. This smooth dependence has its physical meaning: many physical processes are stable with respect to the parameters (except the chaos and critical theory). For the mathematicians, "the door is open to apply the powerful theorems of differential calculus" ([10], e.g. the Implicit Function Theorem).

In the following, we will discuss the continuous dependence (between the solutions and the data) for the parabolic equations with locally arbitrary growth conditions (e.g. Rosseland-type).

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